The Logical Roots of Indeterminacy

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I. Indeterminacy as Relativity to Logical Frameworks

In 1915, Leopold Löwenheim proved a remarkable theorem:

(L) If the domain is at least denumerably infinite, it is no longer the case that a first-order fleeing equation is satisfied for arbitrary values of the relative coefficients. (Löwenheim 1915, p. 235)

In contemporary terminology the theorem says that if a formula Φ of first-order logic with identity is finitely valid but not valid, then for every cardinal $\lambda \geq \aleph_0$, Φ is not λ-valid (i.e., if $\neg\Phi$ is satisfiable in an infinite model, then for every infinite cardinal λ, $\neg\Phi$ is satisfiable in a model of cardinality λ).[1] It follows from this theorem, Löwenheim pointed out, that "[a]ll questions concerning the dependence or independence of Schröder's, Müller's. or Huntington's class axioms are decidable (if at all) already in a denumerable domain." (1915, p. 240).

In a series of articles, Thoralf Skolem (1920, 1922, 1929, 1941, 1958) presented a new version of Löwenheim's theorem and offered a new kind of proof for it. We can formulate Skolem's result as

(LS) Let T be a countable 1st-order theory (where a theory is a set of 1st-order sentences). Then, if T has a model, T has a countable model; in particular: (i) T has a model in the natural numbers; (ii) If $\mathfrak{A}$ is a model of T, then there is a countable submodel $\mathfrak{A}'$ of $\mathfrak{A}$, such that $\mathfrak{A}'$ is a model of T.

Skolem's theorem was extended by Tarski to

(LST) Let T be a set of sentences in a language L of cardinality $\kappa \geq \aleph_0$. Then, if T has an infinite model (a model with an infinite universe), T has a model of cardinality λ for every $\lambda \geq \kappa$.

Skolem regarded LS as signaling the unavoidable relativity of mathematical notions to *logical frameworks*. Skolem's view is sometimes referred to as Skolem's paradox on the basis of passages such as this:

So far as I know, no one has called attention to this peculiar and apparently paradoxical state of affairs. By virtue of the [set-theoretical] axioms we can prove the existence of higher cardinalities, of higher number classes, and so forth. How can it be, then, that the entire domain B [the universe of an "LS model" of set theory] can already be enumerated by means of the finite positive integers? (Skolem 1922, p. 295)

However, the "paradox" is swiftly explained away:

The explanation is not difficult to find. In the axiomatization "set" does not mean an arbitrarily defined collection; the sets are nothing but objects that are connected with one another through certain relations expressed by the axioms. Hence there is no contradiction at all if a set M of the domain B is nondenumerable in the sense of the axiomatization; for this means merely that *within* B there occurs no one-to-one mapping Φ of M onto Z_0 (Zermelo's number sequence). Nevertheless there exists the possibility of numbering all objects in B, and therefore also the elements of M, by means of the positive integers; of course, such an enumeration too is a collection of certain pairs, but this collection is not a "set" (that is, it does not occur in the domain B). (Ibid.)

LS (*LST*) is often viewed as setting a limit to the axiomatic method, and this view is, according to Skolem, correct in a sense:

[A]xiomatizing set theory leads to a relativity of set-theoretic notions, and this relativity is inseparably bound up with every thoroughgoing axiomatization. (Ibid.; p. 296)

[By means of the axiomatic method] the theorems of set theory can be made to hold in a merely *verbal* sense. (Ibid. See also Skolem 1941, p. 468.)

[T]here is no possibility of introducing something absolutely uncountable except by means of a pure dogma. (Skolem 1929, p. 272, translated by Wang 1970, p. 38)

This limitation, however, is nothing more than a price paid for a precise formulation of intuitive notions: In the course of formalizing absolute, yet imprecise, mathematical notions within a formal, i.e., logical, calculus, these notions are inevitably relativized to the logical calculus itself. Thus, Skolem says:

[L]a distinction est essentielle entre la notion simple ou absolue d'ensemble et la notion telle qu'elle découle d'une méthode déterminée de la préciser. La notion étant, dans ce second sens, plus précise et en même temps relative seulement à la façon de la délimiter.[2] (Skolem 1941, p. 480)

The expectation that a formalization will preserve the absolute nature of the intuitive notions is, according to Skolem, unfounded:

Que l'axiomatique conduise au relativisme, c'est un fait parfois considéré comme le point faible de la méthode axiomatique. Mais sans aucune raison. Une analyse de la pensée mathématique, une fixation des hypothèses fondamentales et des modes de raisonements ne peut être qu'un avantage pour la science. Ce n'est pas une faiblesse d'une méthode scientifique, qu'elle ne puisse donner l'impossible.[3] (Ibid., p. 470)

And this impossibility is intimately connected with the idea of a *formal system*:

Mon point de vue est donc qu'on doit utiliser les systèmes formels pur le développement des idées mathématiques. On peut ainsi préciser les notions et les méthodes mathématiques . . . .

Si donc nous désirons avoire une théorie générale des ensembles, cette théorie aussi doit être développée comme un système formel .... Je ne comprends pas pourquoi la plupart des mathématiciens et logiciens ne semblent pas être satisfaits de cette notion d'ensemble définie par un système formel, mais au contraire parlent de l'insuffisance de la méthode axiomatique. Naturellement cette notion d'ensemble a un caractère relatif; car elle dépend du système formel choisi.[4] (Skolem 1958, pp. 634–5)

Tarski adds a twist to Skolem's analysis:

Le théorème de Löwenheim–Skolem lui-même n'est vrai que dans une certaine interprétation des symboles. En particulier si on interprète le symbole $\in$ d'une théorie des ensembles formalisée comme un prédicat à deux arguments analogue à tout autre prédicat, le théorème de Löwenheim–Skolem s'applique, et il existe un modèle dénombrable. Mais par contre si l'on traite $\in$ comme les symboles logiques (quantificateurs, etc.) et qu'on l'interprète comme signifiant appartenance, on n'aura plus en général de modèle dénombrable.[5] (Ibid., p. 638)

Indeterminacy, thus, according to Skolem and Tarski, is relativity to logical frameworks. For Skolem, this relativity concerns the "order" of the logical system involved (*LS* does not hold in full second-order logic) as well as the choice of formalization (axiomatization) within a given logic. For Tarski, this relativity concerns the choice of logical terms: not only does *LST* fail in second-order logic, but it also fails in first-order logic with the membership relation as a logical constant.[6]

II. Indeterminacy as Relativity to Background Language

In a series of influential works, Quine (1958, 1960, 1969a), Putnam (1977, 1978, 1981, 1983a), and others have argued that relativity and indeterminacy are characteristic of language in general, not just of language formalized within a logical framework. Meaning, reference, and ontology are relative to background theory (background language), and a direct, unique, absolute correspondence between words and objects is impossible. A unique correspondence, Putnam says, would require something that goes well beyond our critical resources: something on the order of a *metaphysical dogma.* [Compare with Skolem (1929).]

Indeterminacy as relativity to an external "system of coordinates" was illustrated by a number of suggestive parables. One of the best known of these is Quine's parable of the linguist–explorer: A linguist reaching an unknown land is trying to decipher its inhabitant's discourse. Meaning, however, is underdetermined by linguistic behavior, and it is only by imposing his own scheme of reference (individuation, ontology) upon the native speakers that the linguist is able to create a serviceable manual of translation. In principle, the linguist could use a different conceptual scheme in interpreting the native's discourse, arriving at an empirically equivalent but theoretically divergent manual of translation. The two manuals would agree with the same observational data, but whereas one, for example, would construe 'Gavagai' as a sortal term (or as a statement about discrete objects), the other would construe it as a mass term (or as a statement about undifferentiated "stuff").

Whereas in Quine's parable the background coordinate system is embodied in a member of our own culture (a linguist), in Putnam's the background framework is embodied in a transcendent metaphysical force. Consider the following excerpt from Putnam's parable of "God and the Indeterminacy of Reference":

[*God and the Indeterminacy of Reference.*] [A]t the time of the Tower of Babel episode, God became bored.... Not only did He cause us to start speaking different languages, but He started to play around with the satisfaction relations, the 'correspondences', upon which the words–world connection depends.

To understand what He did, pretend that English was one of the languages in existence back then. Imagine that C_1 and C_2 are two admissible 'correspondences' (satisfaction relations), i.e., that C_1 (respectively, C_2) is the satisfaction relation that one gets if M_1 (respectively, M_2) is the model that one uses to interpret English, where M_1 and M_2 are both models which satisfy all the operational and theoretical constraints that our practice imposes. Then what He did ... was to specify that when a *man* used a word, the word would stand for its image or images under the correspondence C_1, and that when a *woman* used a word, the word would stand for its image or images under the correspondence C_2.

This situation continues to the present day. Thus, there is one set of things – call it the set of *cats* – such that, when a man uses the word 'cat' it stands for that set (in a God's eye view), and a different set of things – call it the set of *cats** – such that when a woman uses the word 'cat' it stands for *that* set (in a God's eye view)....

Notice that the same sentences are true under both of His reference-assignments....

It amused God ... to see men and women talking to each other, never noticing that they were almost never referring to the same objects, properties and relations.... (Putnam 1983a, pp. ix–x)

Indeterminacy, in this parable, is an external feature of language (discourse, theories). Men and women do not realize the indeterminacy of their discourse; God does. God hears Man say: ‡Some cats are black‡,[7] and Woman nodding, ‡Yes, some dogs are white‡; God hears Woman say: ‡I have a white dog‡, and Man echoing: ‡You have a black cat‡. Men and women do not sense anything strange in their dialogue. It is only from God's external, absolute point of view that the meaning (reference, ontology) of cross-gender discourse is indeterminate.

III. Indeterminacy as Loss of Information

Why is relativity to an external framework troubling for philosophers? After all, such relativity does not affect the interaction between language users: from a perspective internal to discourse, meaning, reference, and ontology are perfectly determinate.[8]

Indeterminacy is a meta-theoretical phenomenon, but philosophical theories in general are meta-theoretical in nature. Philosophy seeks to understand the relation between language and the world "from above," so to speak, and it is just at this level of understanding that the impact of indeterminacy is felt. Any discussion requiring both (i) an external point of view on language or theory, and (ii) an unequivocal determination of meaning, reference, or ontology, is made impossible by indeterminacy: the correspondence theory of truth, realistic epistemology, and so forth. From another perspective, indeterminacy adds a new weapon to the skeptic's arsenal: If it is not determined what the statement 'There are indenumerably many stars' says about the world, how can we trust it to tell us "the truth" about it? How can we rely on our theories to give us accurate information about the world if we cannot determine what they say about it? We can view indeterminacy as a barrier to the transmission of information: in theories formulated within the framework of standard first-order logic, a considerable amount of information (e.g., information about the size of large collections) is lost.[9] The idea that indeterminacy is *loss of information* can be illustrated by a sequel to Putnam's parable.

God and the Indeterminacy of Reference – Part II: Many years have passed. God is getting old. God's powers are leaving Him, His perception is deteriorating, His memory is not what it was. At some point God becomes dependent on humankind to provide Him with information about the world. God listens to humans' utterances about the world and updates His ledgers. God hears a man's voice saying: "Some cats are black", and God notes that some cats are black; God hears a woman's voice saying: "Some dogs are black", and God notes that some cats are white. . . .

One day God realizes that He no longer distinguishes between men's and women's voices. God hears a human's voice saying: "Some cats are black", but God does not know whether it is a man saying that some cats are black or a woman saying that some dogs are white. God is now paying a price for His youthful acts. Had He not tampered with Man – Woman communication, He would have known whether in the world some cats are black or some dogs are white. As things stand, He knows that in the world either some cats are black or some dogs are white, but He does not know either that in the world some cats are black or that in the world some dogs are white.

This is Field's 1974 rendition of Quine's indeterminacy thesis. The existence of men – models and women – models amounts to loss of information. God can obtain disjunctive information about the world, but not categorical information. Had God muddled also with children's and adults' use of language, he would have suffered a greater loss of information. We who obtain our information from theories formulated within logical frameworks are worse off than Him.

Discourse formulated within a logical framework is deeply indeterminate. Not only is the meaning of physical terms ('cat', 'black', 'atom', . . .) highly indeterminate in such a discourse, but the meaning of mathematical terms ('number', 'set', 'uncountably many' in the standard logical framework) is also indeterminate. Indeterminacy in logical frameworks can be characterized in terms of "nonstandard" models and denotations: indeterminacy of terms is the existence of *nonstandard denotations*, indeterminacy of theories is the existence of *nonstandard models*. The notions of nonstandard model and nonstandard denotation are relativistic notions. Relative to one criterion of meaning ("intended meaning," "preformalized meaning," "standard meaning," etc.), a given denotation is nonstandard, relative to another – standard. I will call indeterminacy relative to an external standard of meaning 'relative indeterminacy'. Relative indeterminacy within a framework is determined by two things: (i) an external standard of meaning and (ii) the framework's expressive resources. A notion (or a theory) is relatively indeterminate within a given framework if and only if the distinctions required for capturing its "intended" meaning cannot be drawn within that framework. I will call a term's inability to distinguish between referents (or a theory's inability to distinguish between models) 'absolute indeterminacy'. 'Absolute indeterminacy' is a nonrelativistic notion underlying 'relative indeterminacy'.[10]

The difference between relative indeterminacy and absolute indeterminacy can be seen by the following example: Consider the notion 'exactly one', defined in a language L of some logical framework $\mathscr{L}$. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two models for L in $\mathscr{L}$, the first with a universe $A_1 = \{1, 2\}$ and the second with a universe $A_2 =$ {Bill Clinton, George Bush}. The denotation of 'exactly one' in $\mathfrak{A}_1$ is $\{\{1\}, \{2\}\}$, in $\mathfrak{A}_2$ – {{Bill Clinton}, {George Bush}}. Clearly, these two denotations differ one from the other, and in this sense 'exactly one' is absolutely indeterminate in $\mathscr{L}$. 'Exactly one', however, is not relatively indeterminate in $\mathscr{L}$ since its absolute indeterminacy is perfectly compatible with its common meaning. (Relative to that meaning, both denotations are "standard.")

Now consider the concept 'x is president'. Let 'president' be defined by some theory T of L, and let the denotation of 'president' in $\mathfrak{A}_2$ be {Bill Clinton, George Bush}. Then 'president' has a standard denotation in $\mathfrak{A}_2$. Assume $\mathfrak{A}_2$ is isomorphic to $\mathfrak{A}_1$. Then the denotation of 'president' in $\mathfrak{A}_1$ – $\{1, 2\}$ – is nonstandard. The absolute indeterminacy of 'president' leads to relative indeterminacy. The difference between 'president' and 'exactly one' amounts to this: the distinctions required by our external standard of meaning for 'exactly one' *can* be drawn within logical frameworks, whereas those required by our external standard for 'president' *cannot*. We can sum up the distinction between absolute and relative indeterminacy by saying that absolute indeterminacy is the existence of a multiplicity of models (denotations), whereas relative indeterminacy is the existence of nonstandard models (denotations).

To appreciate the depth of indeterminacy in logical frameworks, I will now offer a brief account of absolute indeterminacy in such frameworks.

(Absolute) Indeterminacy in Logical Frameworks

Background Notions

1. *Logical framework.* We regard a logical framework, $\mathscr{L}$, as a pair, $\langle \mathcal{L}, \mathscr{M} \rangle$, or a triple, $\langle \mathcal{L}, \mathscr{M}, \mathscr{P} \rangle$, where $\mathcal{L}$ is a class of formalized languages, $\mathscr{M}$ is a class of apparati of models for languages in $\mathcal{L}$, and $\mathscr{P}$ is a proof system for languages in $\mathcal{L}$. To fix the notion of logical framework, we will restrict ourselves to languages of three kinds: standard first-order languages, generalized first-order languages, and standard higher-order languages, and the corresponding systems of models. In this paper, we disregard $\mathscr{P}$.[11]
2. *Meta-theoretical notions.* The meta-theoretical notions are defined in the usual way. Note in particular the notions of *model for L*, *model of T* (where T is a theory, i.e., set of sentences, in L), and *logical/nonlogical constant of L*.
3. *Term of L.* ξ is *term of L* iff ξ is either a nonlogical constant of L or an open formula of L.
4. *Reference of a term ξ of L in $\mathfrak{A}$.*
   (a) if ξ is a nonlogical constant of L, its reference in $\mathfrak{A}$ is the extension (denotation) assigned to it by $\mathfrak{A}$;
   (b) if ξ is an open formula of L, its reference in $\mathfrak{A}$ is its extension in $\mathfrak{A}$, based on the Tarskian definition of satisfaction-in-a-model for formulae of L.

We will symbolize the reference of the term ξ of L in $\mathfrak{A}$ by $R_{L,\mathfrak{A}}(\xi)$.

5. *Ontology of T.* There are two kinds of theories: (i) theories with a characteristic one-place predicate (primitive or defined) that determines their ontology, (ii) theories without such a predicate. We will say that the *ontological predicate of T* – O_T – is the characteristic predicate of T, if it has one; the self-identity predicate ($x \approx x$), otherwise. The *ontology of T in* $\mathfrak{A}$ is the *referent of O_T in* $\mathfrak{A}$.

Indeterminacy in a logical framework has to do with *variability of reference and ontology under models.* We distinguish four *modes* of indeterminacy and four *types* of indeterminacy.

Modes of indeterminacy. The following four modes have to do with *what is said to be indeterminate*:

(1) *Indeterminacy of terms in L*: variability of the reference of terms of L under models for L.

(2) *Indeterminacy of terms in T*: variability of the reference of terms of L under models of T.
(3) *Indeterminacy of the ontology of T*: variability of the ontology of T under models of T.
(4) *Indeterminacy of T*: indeterminacy of terms in T or indeterminacy of the ontology of T.

Types of Indeterminacy. For each mode, we distinguish four types of indeterminacy, which have to do with *what kind of variations under models* are involved:

(a) *NE-indeterminacy:* variability under non-equivalent models.
(b) *NI-indeterminacy:* variability under nonisomorphic models.
(c) *I-indeterminacy:* variability under isomorphic models.
(d) *A-indeterminacy:* variability under automorphic models.[12]

Combining the mode-type distinctions, we define:

(A) *The* term ξ of L is *NE-/NI-/I-/A-indeterminate in L* iff there are at least two models, $\mathfrak{A}_1$, $\mathfrak{A}_2$ for L such that $\mathfrak{A}_1$, $\mathfrak{A}_2$ are non-equivalent/nonisomorphic/isomorphic/automorphic, and the referent of ξ in $\mathfrak{A}_1$ is different from its referent in $\mathfrak{A}_2$.
(B) *The* term ξ of L is *NE-/NI-/I-/A-indeterminate in T* iff there are at least two models, $\mathfrak{A}_1$, $\mathfrak{A}_2$ of T such that $\mathfrak{A}_1$, $\mathfrak{A}_2$ are non-equivalent/nonisomorphic/isomorphic/automorphic, and the referent of ξ in $\mathfrak{A}_1$ is different from its referent in $\mathfrak{A}_2$.
(C) *The ontology of the theory T* is *NE-/NI-/I-/A-indeterminate* iff there are at least two models, $\mathfrak{A}_1$, $\mathfrak{A}_2$, of T such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are non-equivalent/nonisomorphic/isomorphic/automorphic, and the ontology of T in $\mathfrak{A}_1$ is different from its ontology in $\mathfrak{A}_2$.
(D) *The theory T is NE-/NI-/I-/A-indeterminate* iff either some term of L is *NE-/NI-/I-/A*-indeterminate in T or the ontology of T is *NE-/NI-/I-/A*-indeterminate.

Definition (D) is reducible to.

(D′) The theory T is *NE-/NI-/I-/A-indeterminate* iff some term of L is *NE-/NI-/I-/A*-indeterminate in T.

Centering our attention on *theory* indeterminacy, we note the following theses:

(I) *Thesis of NE-indeterminacy:* A theory is *NE*-indeterminate iff it is incomplete.
(II) *Thesis of NI-indeterminacy:* Every theory with an infinite ontology is *NI*-indeterminate in the standard first-order framework.

(III) *Thesis of I-indeterminacy:* Every consistent theory is I-indeterminate.

(IV) *Thesis of A-indeterminacy:* A theory T is A-indeterminate iff there is at least one model $\mathfrak{A}$ of T and at least one term ξ of L such that the referent of ξ in $\mathfrak{A}$ is not closed under all permutations of the universe of $\mathfrak{A}$. (Keenan forthcoming.)[13]

It follows from these theses that every consistent theory formulated within the framework of standard first-order logic (with identity) suffers loss of information: every incomplete theory suffers loss of information about truth; every theory admitting an infinite ontology suffers loss of information about quantity; every (consistent) theory whatsoever suffers loss of information about the identity of its ontology; and every theory with at least one name and an ontology of cardinality larger than 1 suffers loss of information about "who is who" within the said ontology. Thus, take for example, first-order Peano arithmetic: Peano arithmetic fails to determine (i) the truth of some arithmetic statements, (ii) the size of the class of natural numbers, (iii) the identity of the natural numbers [what kind of objects the natural numbers are: von Neumann sets, Zermelo sets, some other kind of object (Benacerraf 1965)], and (iv) what element plays the role of what natural number in what ontology. The loss of information involved in formulating arithmetic within the framework of standard first-order logic is large indeed.

Indeterminacy, however, is loss of information in a more intricate way than suggested so far. Suppose we formulate Peano arithmetic in a first-order language with the logical quantifier 'there are exactly $\aleph_0$ x such that. . . .'[14] Within that framework, Peano arithmetic can easily be expanded to a theory that gives us precise information on the quantity of natural numbers: If we add to Peano arithmetic the axiom '$(\aleph_0 x)x \approx x$', all models of the expanded theory will contain exactly $\aleph_0$ elements. Using the new framework will not free Peano arithmetic from all its indeterminacies, but its NI-indeterminacy will be significantly reduced. In particular, the "new" arithmetic will have no models of cardinality larger than $\aleph_0$.

Indeterminacy, then, comes in degrees: the stronger (the more expressive) a given logical framework is, the weaker the indeterminacy it generates. But indeterminacy is not restricted to logical frameworks. Logical frameworks are not the only kind of framework, and relativity to different kinds of framework gives rise to different kinds of indeterminacy, hence to loss of different kinds of information. *Logical* frameworks are marked by loss of *extralogical* information, *experiential* frameworks (frameworks fixing the experiential content of various terms), by loss of *theoretical*, including *logical*, information. Much of the philosophical literature on indeterminacy is concerned with relativity to *logical* frameworks. In Putnam's parable, for example, God mixes up men's and women's understanding of *extralogical terms*, not their understanding of *logical terms*. The *logical* content of human discourse does not vary from gender to gender, only its *extralogical* content does. Likewise, in logical frameworks, *extralogical* content varies from model to model; *logical* content is uniquely determined by the logical framework.

Logical frameworks, however, are (as we have noted) not the only background frameworks. An example of discourse conducted within a different kind of framework – an *experiential* framework – is found in Quine's (imaginary) case study of the field linguist.[15] Spotting a rabbit, the natives (consistently and repeatedly) utter the word (sentence?) 'Gavagai'. The linguist, who shares their observational standpoint, jots 'rabbit?' in his manual. The question mark conveys his conundrum regarding the individuation criterion associated with 'Gavagai': Is 'Gavagai' a word for discrete rabbits? Rabbit stuff (undetached rabbit parts)? As long as the linguist's framework is purely experiential, the individuative status of 'Gavagai' is indeterminate. His triangular experience of observing a rabbity thing,[16] observing the natives observing a rabbity thing, and observing the natives uttering 'Gavagai', does not suffice to individuate the referent of 'Gavagai'. Individuation, Quine notes, is a matter of *logical* parameters: identity, quantification, Boolean operations. But no amount of *experiential evidence* will determine the logical parameters of the native's utterances. Suppose the linguist queries: 'Is this Gavagai the same as that?' A positive answer to this question will not adjudicate the matter. The import of such an answer depends on the native's interpretation of the linguist's question, and as far as the purely experiential data available to the linguist go, the native may be interpreting the question as: 'Does this Gavagai belong with that?' ('Is this Gavagai part of the same stuff as that?') (Quine 1969a, p. 33) The problem is not evidential; the problem is factual. There is simply *no experiential fact of the matter* as to whether a stimulus generated by a rabbit sighting is a stimulus of a discrete rabbit or a stimulus of rabbit stuff. Introducing the idea of *experiential models* (models preserving the experiential features of a given discourse), we can say that *stimulus meaning* (meaning as determined by experiential stimuli) is fixed throughout experiential models, but logical parameters – hence individuation – are not. In all experiential models, 'Gavagai' denotes a rabbity thing, but in some experiential models, it denotes discrete rabbits whereas in others, undetached rabbit parts; in some experiential models 'is the same as' functions as identity; in others, as the relation of belonging with. The *logic* of purely experiential discourse is underdetermined.

Indeterminacy is loss of information: loss of information about who is who, who possesses what properties, who stands in what relations to whom, and so on. Indeterminacy as loss of information is a universal predicament of *partial* frameworks: frameworks that "fix" the logical parameters of discourse but not its observational parameters, frameworks that fix its observational parameters but not its logical parameters, frameworks that fix its logical and observational parameters but not its theoretical parameters,[17] and so on. To the extent that human discourse is commonly conducted within partial frameworks, indeterminacy is a universal predicament of human discourse.

Indeterminacy, we have seen, is a barrier to knowledge, and our account explains why this is so: knowledge means an increase in information, but indeterminacy means loss, or dilution, of information. Indeterminacy, however, is not just a negative element in the generation of knowledge. Indeterminacy, like other phenomena of human cognition, plays a positive as well as a negative role in producing knowledge and information.

IV. Indeterminacy as Specificity of Information

To see what positive role indeterminacy plays in the production of information, let us turn once again to Quine:

> Imagine a fragment of economic theory. Suppose its universe comprises persons, but its predicates are incapable of distinguishing between persons whose incomes are equal. The interpersonal relation of equality of income enjoys, within the theory, the substitutivity property of the identity relation itself; the two relations are indistinguishable. (1969a, p. 55)

Quine's economic theory in effect generates a framework in which individuals with equal economic attributes are not differentiated. But the absence of a more discriminating apparatus of individuation does not detract from the efficiency of the theory. On the contrary: to discriminate individuals according to economically irrelevant features – for example, hair color, hour of birth, favorite movie star – would only introduce clutter into the theory, obscure its content, and decrease its efficiency. Given its goal, the theory's nonstandard method of individuation is well motivated: its identity principle is not undiscriminating; it is tailored to the needs of a highly specialized theory. Indeterminacy, as far as our economic theory is concerned, is *specificity* of information rather than *loss* of information.

The shift from loss of information to specificity of information can be explained as follows: Indeterminacy, in its most general form, is *partiality* of information, but partiality of information is both the presence and the absence of information. Indeterminacy occurs as an element in a pair: *indeterminacy* as the loss or absence of information, complemented by *determinacy*, the presence and specificity of information. Quine's economic theory does not distinguish between individuals with different hair colors (indeterminacy), but does distinguish individuals with different income levels (determinacy); the native's utterance fails to inform us of the identity of that which has passed by an instant ago (indeterminacy), yet by uttering 'Gavagai' the native has succeeded in conveying to us a very specific bit of information, namely, that that which has just passed by is a *rabbity-thing* (rather than an *elephanty-thing* or a *snaky-thing* or ...) (determinacy). The meaning of 'Gavagai' is partially determinate, partially indeterminate, and it is only by sorting out its determinate elements from its indeterminate ones that its net informative value can be calculated.

Relative indeterminacy is doubly relative: relative to an external standard of meaning, and relative to a task at hand. Relative to one task (e.g., the task of generating a Native-English manual of translation), indeterminacy of individuation is loss of information; relative to another (the task, say, of signaling the presence of edible things), it is not. Relative to some goal, failure to distinguish individuals with different hair color is loss of information; relative to another – specificity of information. To view indeterminacy as specificity of information is to view it as a tool for the generation of determinacies. We can illustrate the positive role that indeterminacy plays in knowledge by a new parable of "God and His Pursuit of Knowledge":

The New Parable of Indeterminacy. When God was young, His interests were all-encompassing. God was interested in every detail of every happening in every corner of the world: whose cat was black, whose dog was white, which pebble lay on what riverbed, and so forth. As God grew older, His interests became refined. Today, God's interests are restricted to the universal laws of nature. And God knows that someday His interests will reach the pinnacle of purity. Someday God will arrive at the age of wisdom, and from that day on His interests will focus on the logical structure of the world. In preparation for this day and the obvious limitations associated with old age, God is spinning a clever plot. We can sum it up by one maxim: Let each human speak his/her own language, but let all use the same logic. God reasons as follows: If all humans assign the same reference (interpretation) to the logical particles of their language but differ in the assignment of reference to the nonlogical particles, then, by sifting through the common elements of their discourse, one could find what logical features they attribute to the world. So God decrees: Let all humans use the same syntax and let them all use the same semantic rules for the logical constants of their language. But let each human use his/her own ontology and his/her own scheme of nonlogical reference. Listening to humans talk, God will hear a human say: ‡Snow is white *or* snow is *not* white‡, and the reverberations of his/her utterance: ‡Grass is green *or* grass is *not* green‡, ‡Sand is blue *or* sand is *not* blue‡, ... And God will know that, given a universe X, an object y *in* X, and a subset Z of X, y is in the union of Z and its complement in X. In symbols: $y \in ZU(X - Z)$. God will have learnt, or will have relearnt, the ontological version of the law of excluded middle. This is God's plan for obtaining logical information. Had God been interested in obtaining physical information, He would have "fixed" the physical constants of human language. Since His plan is to obtain logical information, he is "fixing" its logical constants.[18]

Extralogical indeterminacy is a means of extracting logical information. Logical frameworks are designed to convey a special type of information, namely, logical information; therefore the logical structure of statements and theories formulated within them are fully determinate (relative to their intended meaning), but their extralogical content is not. Extralogical variation is a tool for identifying logical regularities: logical truths (laws), logical consequences, logical consistencies, logical equivalences, and so on. Whether a given indeterminacy is loss of information or specificity of information is thus relative to context:

relative to God's youthful aims, indeterminacy of extralogical vocabulary is loss of information; relative to his golden-age aims – specificity of information.

Our claim that logical frameworks are intrinsically limited to the transmission of logical information challenges a widely held view on the utility of logical frameworks.

V. Does Logic Provide a General Framework for the Construction of Theories?

What is the role of logical frameworks in the development of theories (the transmission of information[19])? Since the birth of modern logic over a century ago, two competing views have emerged. According to one view (the "generalist" view), logic provides a *general, all-purpose* framework for the construction of theories, designed to improve their overall conceptual clarity, enhance their descriptive as well as explanatory capabilities, and increase their predictive power (where applicable). According to the second view (the "specialist" view), logic provides a *special* framework for the formulation of theories, designed to facilitate the discovery of their (distinctively) *logical* consequences, their *logical* consistency or inconsistency, the *logical* dependence or independence of their axioms, and so on. An early locus of both approaches is Frege's (1879) *Begriffsschrift*, where the idea of an artificial symbolic notation is justified on the basis of two kinds of considerations: general "Leibnizian" considerations stressing the need for a universal tool for the precise expression of ideas, and special logical (or, rather, meta-logical) considerations calling for the construction of a specialized device for carrying out logical proofs.

The general benefits of a symbolic notation were emphasized by Frege in the preface to his monograph. Frege opened *Begriffsschrift* with general methodological considerations:

> In apprehending a scientific truth we pass, as a rule, through various degrees of certitude. Perhaps first conjectured on the basis of an insufficient number of particular cases, a general proposition comes to be more and more securely established by being connected with other truths through chains of inferences.... Hence we can inquire ... how we can finally provide it with the most secure foundation. (Ibid., p. 5)

A secure foundation for knowledge cannot be provided within natural language, however, due to its *imprecision* of expression. A more precise notational system is required, one related to natural language as a scientific instrument is to the "naked" eye.

> I believe that I can best make the relation of my ideography to ordinary language clear if I compare it to that which the microscope has to the eye.... as soon as scientific goals demand great *sharpness* of resolution, the eye proves to be insufficient. The microscope, on the other hand, is perfectly suited to precisely such goals [Ibid., p. 6].[20]

Frege likened his idea of a symbolic ideography to Leibniz's idea of a universal symbolism: "Leibniz, too, recognized ... the advantages of an adequate system of notation. His idea of a universal characteristic, ... a *calculus philosophicus* or *ratiocinator*" was a "worthy goal" which, if realized, would lead to an "immense increase in the intellectual power of mankind." (Ibid.)[21] Even the name that Frege gave to his notation suggests a general goal: 'Begriffsschrift', "a notation of ideas," rather than 'Logikbegriffsschrift', "a notation of *logical* ideas."

Another line of thought developed in the preface, however, points to a narrower conception of Frege's symbolism. Among the statements emphasizing the *logical* nature of Frege's ideography are the following:

> [W]e divide all truths that require justification into two kinds, those for which the proof can be carried out purely by means of *logic* and those for which it must be supported by facts of experience.
>
> The most reliable way of carrying out a proof, obviously, is to follow pure *logic*.
>
> [The] first purpose [of the proposed ideography] is to provide us with the most reliable test of the validity of a *[logical] chain of inferences*.
>
> [We exclude from this ideography] anything that is without significance for the *[logical] inference sequence*. (Ibid., pp. 5–6, Frege's emphasis)

It is quite clear from these statements that Frege conceived his *Begriffsschrift* as a specifically *logical*, rather than "universal," language. Moreover, it follows from Frege's Logicist Project – the project of reducing mathematical knowledge to *purely logical* knowledge – that the language in which mathematical theories are to be reconstructed is a *purely logical* language. Finally, Frege himself drew a sharp distinction between a "purely logical system" and a "universal system of notation." The two systems, according to Frege, differ in their treatment of objects: "*pure logic ... disregard[s] the particular characteristics of objects*," while *a Leibnizian "system of notation [is] directly appropriate to objects themselves*" (Ibid., pp. 5–6, my italization). Frege explained the difference between a purely logical language and a general Leibnizian language by differences in their goals: logical languages are designed to express a special kind of laws whereas Leibnizian languages are designed to express laws of objects in general. More specifically, logical languages are intended to express those "laws upon which *all* knowledge rests" and which therefore "transcend all particulars" (Ibid., p. 5); Leibnizian languages are intended to express the whole gamut of laws constituting our knowledge, including laws applicable to objects directly and in their particularity.

The Leibnizian, or generalist, approach to logical frameworks is exemplified in Skolem's explanation of the utility of logic. Starting with intuitive and vague mathematical (or scientific) ideas, we use the resources of modern logic to generate sharply delineated "images" (representations, counterparts) of these ideas. For example, by axiomatizing set theory within the framework of standard first-order logic, we are able to define the intuitive set-theoretical notions of 'function', 'ordinal number', 'cardinal number', 'finitely many', 'uncountably many', and so on in a sharp and precise manner, reducing them to the membership relation whose properties are specified by the axiomatized theory.

The relative indeterminacy of extralogical notions within logical frameworks, however, challenges Skolem's approach. Skolem regarded indeterminacy as the price we pay for increasing the precision of our (extralogical) concepts, but the price we pay is, in effect, a *decrease* in their precision. How can the axiomatization of set theory within the framework of standard first-order logic be said to yield a precise notion of uncountability if any consistent statement of the form 'S is uncountable' is satisfied by a countable model? And how can a first-order formulation of number theory be said to yield a precise notion of natural number, if the quantity (not to say the identity) of objects falling under it is highly indeterminate?

We can explain the failure of logical frameworks to transmit accurate nonlogical information by reference to "Frege's principle": it is because logical frameworks do not distinguish 'the particular characteristics of objects' (see above) that notions based on such characteristics cannot be accurately formulated within these frameworks. A contemporary version of Frege's principle is the invariance principle for logical constants:

(*LI*) *Logical Invariance:* Logical constants are invariant under isomorphic argument-structures. (Mostowski 1957, Lindström 1966, Tarski 1966, Sher 1991, and others)

This principle says that if C^n is a logical constant (a logical predicate, quantifier, or function) of a language L of a logical framework $\mathscr{L}$, and $\langle A, \beta_1, \ldots, \beta_n \rangle$ is an argument-structure for C^n – namely, a structure consisting of a universe A followed by n elements of types corresponding to those of the arguments of C^n – then,

If $\mathfrak{A}$ and $\mathfrak{A}'$ are models for L (in $\mathscr{L}$) with universes A and A', respectively, and the argument-structures $\langle A, \beta_1, \ldots, \beta_n \rangle$ and $\langle A', \beta'_1, \ldots, \beta'_n \rangle$ are isomorphic, then $\langle \beta_1, \ldots, \beta_n \rangle$ satisfies C^n in $\mathfrak{A}$ iff $\langle \beta'_1, \ldots, \beta'_n \rangle$ satisfies C^n in $\mathfrak{A}'$.[22]

The intuitive meaning of (*LI*) is that logical constants do not distinguish formally identical objects, or, logical constants discern only formal patterns of objects possessing properties and standing in relations (not their "material" features). Since the expressive power of logical frameworks is largely determined by their logical constants, it follows from (*LI*) that the expressive capability of logical frameworks is restricted to the formal.[23] This restriction is captured by the *thesis of logical I-indeterminacy*:

Extend the notion of term of L to include logical constants of L and extend the notion of 'reference in $\mathfrak{A}$' accordingly.[24]

The Thesis of Logical Indeterminacy. Let $\mathscr{L}$ be a framework satisfying (*LI*), L a language in $\mathscr{L}$, T a theory in L, and ξ a term of L. Then:

(i) ξ is I-indeterminate in L;
(ii) ξ is I-indeterminate in T;
(iii) the ontology of T is I-indeterminate;
(iv) T is I-indeterminate.

Although logical frameworks do not allow the precise expression of nonformal notions, formal notions are, in principle, accurately expressible in such frameworks. This fact is (partly) reflected in the absolute A-determinacy of such notions.

Define 'logical notion' as:

ξ is a *logical notion* of a language L in a logical framework $\mathscr{L}$ iff ξ is either a logical constant of L or an open formula of L with no nonlogical constants.

The Principle of Logical Plenum. If ξ is a logical notion of a language L in a logical framework $\mathscr{L}$, then ξ has exactly the same reference in all automorphic models for L, i.e., ξ is A-determinate.

This principle says that the extension of logical notions in any given model is formally "full" in the sense of:

Closure under Permutations. A notion (term) ξ of L is A-determinate in L/T iff for any model $\mathfrak{A}$ for $L/$ of T with universe A, the reference of ξ in $\mathfrak{A}$ is closed under all permutations of A. (Lindenbaum and Tarski 1934–5, Sher 1991, Keenan forthcoming, and others. See Thesis IV, above.)[25]

It is characteristic of formal, or mathematical, notions in general that they can be so formulated as to satisfy the principle of logical plenum and, more generally, the principle of logical invariance (*LI*). For that reason logical frameworks are naturally suited for the expression of formal as well as meta-formal ideas, that is, the ideas of formal law (truth), formal consequence, and so on.[26]

Our analysis of logical frameworks is in the spirit, if not in the letter, of Frege's narrower conception. A logical framework is an instrument designed for a particular purpose. Its primary task is to identify the logical properties and relations of theories, and to this end it is tuned to those features of theories (their referents, ontology) that are relevant to the logical task but not to others. The extralogical information transmitted by a logical framework is largely indeterminate, but the logical skeleton of that information is highly determinate. The logical skeleton of a piece of information is, however, itself a piece of information; therefore, a logical framework can be viewed as a tool for the transmission of *logical information*.

Turning back to Skolem and the axiomatization of set theory, I would say that what this axiomatization achieves is not a general sharpening of the set-theoretical notions (the indeterminacy of '$\in$' is hardly a sign of sharpness), but rather a sharpening of the logical kernel of these notions. Whether a given indeterminacy means loss or specificity of information (or neither) is largely a matter of what the framework is designed to accomplish. Relative to the "standard" conception of logical consequence (a conception according to which standard first-order logic fully captures the intended notion of logical consequence) the indeterminacy of 'uncountably many' is *not* loss of information (since it does not impede the derivation of any logical consequence), relative to other conceptions [e.g., Sher (1991, 1996a)] it is.

A logical framework is, in general, not an all-purpose framework for the construction of theories, yet sometimes a logical framework is so aligned with a given (preformalized) theory that it is possible to fully express the theory's content by purely logical means. When such an alignment occurs, we say that, for this theory and that logical framework, the *logicist project* is realized. The Logicist Thesis constitutes a bridge between the logical and the Leibnizian projects. Since Frege's goal was to capture the content of mathematical (or, more narrowly, arithmetical) concepts by purely logical means, his language was designed to be at once a logical language and a general language for the expression of mathematical ideas. But even so, Frege's *Begriffsschrift* is inherently logical: it only due to the logical nature of arithmetical notions (according to Frege's position) that *Begriffsschrift* can serve as a general framework for the construction (or reconstruction) of arithmetic.

VI. Full Determinacy as the Absence of Knowledge

Is it possible to construct an altogether general framework for the formulation of theories, a framework in which their logical, experiential, and theoretical constituents are all uniquely determined? Contemplating the possibility of realizing Leibniz's ideal, Frege says:

> The enthusiasm that seized [Leibniz] when he contemplated the ... system of notation [he envisaged] led him to underestimate the difficulties that stand in the way of such an enterprise. But, even if this worthy goal cannot be reached in one leap, we need not despair of a slow, step-by-step approximation.... It is possible to view the signs of arithmetic, geometry, and chemistry as realizations, for specific fields, of Leibniz's idea. The ideography proposed here adds a new one to these fields, indeed the central one, which borders on all the others. If we take our departure from there, we can with the greatest expectation of success proceed to fill the gaps in the existing formula languages, connect their hitherto separated fields into a single domain, and extend this domain to include fields that up to now have lacked such a language.
>
> I am confident that my ideography can be successfully used ... when the foundations of the differential and integral calculus are established.

> It seems to me to be easier still to extend the domain of this formula language to include geometry. We would only have to add a few signs for the intuitive relations that occur there. In this way we would obtain a kind of *analysis situs*.
>
> The transition to the pure theory of motion and then to mechanics and physics [where "besides rational necessity empirical necessity asserts itself"] could follow at this point. (Frege 1879, pp. 6–7)

Considering Frege's program in the present context, we can distinguish four ways of transforming a given logical framework into a general conceptual framework. The first two methods have to do with axiomatization of theories within the framework, the last two with adding new "distinguished" constants to the framework (i.e., new constants whose "intended" interpretation is "hardwired" into the framework). The four methods are: (i) axiomatizing theories within the framework, (ii) specifying an intended model (or models) of axiomatized theories, (iii) adding new logical constants to the framework, (iv) adding new extralogical distinguished constants (and making appropriate adjustments in the apparatus of models).[27] Each of these methods has its uses, but each also has its limitations. We have already noted the limitations of the first method. The second method renders the axiomatic method (as a method for capturing the exact content of theories) redundant: if it is possible to single out a model (which, from the point of view of the axiomatization, is indistinguishable from a host of other models), as capturing the precise content of a given theory, the axiomatization itself is superfluous. The third method does lead to a considerable gain in the expressive capabilities of the framework, but this gain is, as we noted earlier, limited to formal notions. Cardinality statements can be expressed with full precision and determinacy, but physical statements cannot.

The fourth method amounts to adding a new layer to the initial logical framework, that is, combining the logical framework with one or more other frameworks, for example, a theoretical physical framework, an experiential framework, or an everyday objectual framework. (A physical framework has physical distinguished constants satisfying a principle of physical invariance[28] and an apparatus of models representing all physically possible structures of objects relative to a given language.) The "layering" method is familiar from other contexts. To design an artifact, for example, an airplane, we integrate a number of scientific theories into a single application guide. Likewise, to arrive at a unique interpretation (unique model, unique reference, etc.) of a real-life discourse or a real scientific theory, we integrate various conceptual frameworks into a single whole. The new conceptual framework treats all constants (or rather, all undefined constants) as distinguished, eliminating relative indeterminacy and zeroing in on a "standard" model. Here, singling out a model is not an act of "deus ex machina"; rather, the selection of models is based on a set of background guidelines brought together deliberately by the combination method.

The combined framework, however, is parasitic upon the constituent frameworks. Just as an applicational system in science validates, rather than cancels, the independent existence of the constituent theories, each accounting for some specific aspect of nature and overlooking all others, so an applicational framework in semantics mandates the existence of the partial constituent frameworks, each designed for the determinate expression of some notions, some elements of theories, but not others. A single model, a single referent, means absolute particularity, but knowledge requires some degree of generality, hence some degree of indeterminacy. Consider, once again, logic. Not only does metalogical knowledge (e.g., knowledge of what follows logically from what) require the existence of a broad array of models and a broad array of referents of extralogical notions (i.e., a high degree of extralogical indeterminacy, including relative indeterminacy), but the logical notions themselves obtain their meaning through the abundance of models and referents (i.e., through the indeterminacy of their extralogical counterparts). Take primitive logical notions, that is, logical constants, first. The standard logical constants are relatively determinate within the standard logical framework, but their relative determinacy involves absolute indeterminacy: it follows from (*LI*) that the standard logical constants are at least I-indeterminate, and in fact, the standard logical constants are also *NI*-indeterminate. Thus, take the extension (reference) of '$\exists$' in two nonisomorphic models, $\mathfrak{A}_1$ and $\mathfrak{A}_2$, whose universes are $\{a\}$ and $\{a, b\}$, respectively, $a \neq b$. The extension of '$\exists$' in $\mathfrak{A}_1$ is $\{\{a\}\}$, and its extension in $\mathfrak{A}_2$ is $\{\{a\}, \{a, b\}\}$; obviously the two extensions are not equal. In a similar way we can show that '$\exists$' is *NE*-indeterminate. It is only in terms of A-determinacy (the Plenum principle) that the standard logical constants are absolutely determinate. The absolute indeterminacy of the logical constants extends to logical notions in general. 'Exactly one' is I-, *NI*-, and *NE*-indeterminate in the absolute sense, just like '$\exists$' ('at least one') and '$\approx$' ('is identical to').

The absolute indeterminacy of the logical notions, however, does not involve the loss of logical information. On the contrary: the pattern of indeterminacy of a given logical notion constitutes its meaning. The meaning of identity is a pattern across models (the pattern '$\langle\langle a_1, a_1\rangle, \langle a_2, a_2\rangle, \ldots, \langle a_\alpha, a_\alpha\rangle, \ldots\rangle$', where '$a_1$', '$a_2$', ..., '$a_\alpha$', ... represent members of the universe of an arbitrary model); the meaning of the existential quantifier is another pattern across models; the meaning of 'exactly one' is a third pattern, and so on and so forth. The logical laws delineate another constant pattern across models. The pattern displayed by the law of the excluded middle – $\forall x(\Phi x \vee \neg\Phi x)$ – consists, as we have seen, in the universality of a union: the union of any subset of a given ontology with its complement in that ontology. The pattern displayed by the law of noncontradiction – $\neg(\exists x)(\Phi x \ \& \ \neg\Phi x)$ – consists of the emptiness of an intersection: the intersection of any subset of a given ontology and its complement in that ontology. We can characterize a logical law as determinacy bounded by indeterminacy: the determinacy of the pattern represented by $\neg\exists x(- \ \& \ -)$ against the indeterminacy of the pattern represented by Φx.

A logical law is a path across a field of indeterminacy. A physical law is a different kind of path, across a different field of indeterminacy. Full indeterminacy is the absence of knowledge, but so too is full determinacy. Knowledge is a network of determinacies against a background of indeterminacies. To generate a concept is to abstract from something (to overlook something). To draw a pattern is to relegate some details to the background. We can see a shooting star in the darkness of night, but not in the brightness of daylight. . . .

NOTES

The impetus for this paper came from Parsons' comments (in conversation) on the interest of Quine's indeterminacy thesis and his numerous observations on the interrelations between logic, ontology, and language. See, for example, Parsons (1965, 1971, 1982, 1983a,b). An earlier version of this paper was read to the Workshop in the Philosophy of Logic and Mathematics at the University of California at Irvine. I am thankful to the participants for insightful comments. I also thank Peter Sher for comments and advice.

1. More literally, the theorem says: Given a formula ("equation") Φ of the first-order calculus of relations ("relatives") with identity such that Φ is finitely valid but not valid (Φ is a "fleeing equation"), then: if D is a domain of objects of *any* infinite cardinality (D is "*at least* denumerably infinite"), Φ is not valid in D ["it is no longer the case" that Φ is "satisfied for arbitrary values of its relative coefficients" (its relational symbols) in D]. That is, if $\neg\Phi$ is satisfiable in *any* infinite domain, it is satisfiable in *every* infinite domain.
2. Free translation:

   "The distinction between the simple or absolute notion of set and the notion obtained by a determinate method for making the former notion precise is essential. The second notion is more precise but, at the same time, is relative to the manner in which it is delimited."

3. Free translation:

   "The fact that axiomatization leads to relativism is sometimes considered the weak point of the axiomatic method. But without reason. Analysis of mathematical thought, determination of fundamental hypotheses, and modes of reasoning, are nothing but an advantage for the science in question. It is not a weakness of a scientific method that it cannot do the impossible."

4. Free translation:

   My point of view is, then, that we ought to use formal systems for the development of mathematical ideas. In this way we will be able to render the mathematical notions and methods precise. . . . If, then, we wish to have a general theory of sets, this theory should also be developed as a formal system. . . . I do not understand why most mathematicians and logicians seem to be unsatisfied with such a notion of set defined by a formal system, but on the contrary speak about the insufficiency of the axiomatic method. Naturally this notion of set has a relative character: it depends on the chosen formal system.

5. Free translation:

> The Löwenheim – Skolem theorem itself is true only within a certain interpretation of the symbols. In particular, if we interpret the symbol $\in$ of a formalized set theory as a two-place predicate analogous to any other [nonlogical] predicate, the Löwenheim–Skolem theorem applies and there exists a denumerable model. But on the other hand, if we treat $\in$ like a logical symbol (quantificational etc.) interpreted as signifying membership, we will, in general, not have a denumerable model.

6. For a semantic account of what it means to treat a constant as logical, see Sher (1991, 1996a, 1996b). Tarski himself (in his 1966 lecture) regarded the higher-order, but not the first-order, membership relation as an admissible logical constant.
7. Text enclosed in "‡" represents God's point of view. The relevant instances of Putnam's star (*) mapping should be obvious from the context.
8. Of course language still suffers from well-known problems of ambiguity: homonymy, amphibology, and so on, but these do not concern us here.
9. (a) Here and later, my general statements apply to reasonably rich languages. Thus, suppose Quine's linguist seeks to translate the natives' "Gavagai" to a language with no logical (hence no individuative) terms. In a translation to such a language, indeterminacy may not arise. (b) The present chain of reasoning is challenged in Sher [1998/99] on the basis of considerations developed in Sections IV–VI. (c) By "standard first-order logic," I mean a system of logic similar to those presented in most textbooks of mathematical logic [e.g., Enderton (1972)]. The adjective "standard" is intended to connote, among other things, the traditional choice of logical constants in such systems.
10. Absolute indeterminacy is, of course, relative to choice of framework.
11. For generalized first-order languages, see Mostowski (1957), Lindström (1966), Barwise and Feferman (1985), and Sher (1991).
12. Models $\mathfrak{A}$, $\mathfrak{B}$ for L with universes A, B, respectively, are non-equivalent iff for at least one sentence σ of L, the truth value of σ in $\mathfrak{A}$ is different from its truth value in $\mathfrak{B}$. $\mathfrak{A}$, $\mathfrak{B}$ are isomorphic iff there is at least one 1-1 function from A onto B that preserves functions and relations in $\mathfrak{A}$. $\mathfrak{A}$, $\mathfrak{B}$ are automorphic iff $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic and $A = B$. *Note:* The thesis and characterizations based on the notion of *NE*-indeterminacy are less pertinent for my discussion than are those based on *NI*-, *I*-, and *A*-indeterminacy. [The idea of treating *NE*-indeterminacy as a special case of indeterminacy appears in Hansen (1987).]
13. When the extension of ξ in $\mathfrak{A}$ is a subset of A^n or a function from A^n to A, we mean by '$R_{L,\mathfrak{A}}(\xi)$ is closed under a permutation p of A' that $R_{L,\mathfrak{A}}(\xi)$ is closed under the permutation p^* of $\mathscr{P}(A^n)$ – the power set of A^n – or $\mathscr{P}(A^n \times A)$, induced by p'.
14. See references in note 11.
15. (a) There are two ways of approaching Quine's case study as an example of indeterminacy: (i) the linguist himself detects the indeterminacy of the natives' discourse; (ii) it is we, the observes, who detect the indeterminacy of the linguist's understanding of the natives' discourse; the linguist is part of the observed situation. In the present construal, I adopt (i), but this choice is not essential for my point.
(b) The present construal of the 'Gavagai' indeterminacy as representing loss of information is, of course, offered as a *new* interpretation of Quine's "case study" rather than as a *neutral* report of it.
16. I use 'a rabbity thing' as an individuation-wise neutral expression, that is, an expression that does not distinguish between discrete rabbits and rabbit stuff.

17. We may view some of Putnam's discussions of indeterminacy as relating to background frameworks of this kind.
18. This parable should to be taken with a grain of salt (i.e., as a parable rather than as a foolproof method for determining the logical structure of the world). For example, we did not take into account human fallibility, we assumed human language is rich enough and the number of people large enough to cover all formally possible unions of sets and their complements, we assumed God is not subject to the limitations of ω- (and higher) incompleteness, and so on.
19. In this paper, I treat knowledge essentially as information.
20. (a) Here and in later citations, the emphasis is mine (unless otherwise indicated).
(b) The microscope analogy can be interpreted either as supporting a Leibnizian conception of logical languages or as supporting a specialist conception of such languages. In the first case, we view the microscope as an instrument for observing small things in general; in the second – as an instrument for observing things of a special kind.
21. The belief that a universal symbolic language would lead to an "immense increase in the intellectual power of mankind" is attributed by Frege to Leibniz. But Frege himself appears to endorse this belief.
22. To apply (*LI*) to connectives as well as to functions and predicates, we can either add a special entry saying that logical connectives are invariant under identical truth-structures or we can construe the connectives as designating set-theoretical operators: '$\neg$', complement; '&', a family of Cartesian product operators, including intersection (since $A \cap B = \{a : \langle a, a\rangle \in A \times B\}$). [In this connection, see Lindström (1966) and Sher (1991, 1996a,b).]
23. By 'formal' in this paper, I mean 'formal in a semantic rather a syntactic sense'. See Sher (1996a).
24. Given a model $\mathfrak{A}$ with a universe A, the referent of '$\approx$' in $\mathfrak{A}$ is $\{\langle a, a\rangle : a \in A\}$, the referent of '$\exists$' in $\mathfrak{A}$ is $\{B \subseteq A : |B| > 0\}$, and the referents of the truth-functional connectives are based on their analysis either as truth-functional operators (in which case models will be assigned two distinguished elements, T and F) or as set-theoretical operators. (See note 22.) The referent of an open formula containing no nonlogical constants is its extension in $\mathfrak{A}$ based on the Tarskian definition of satisfaction in a model.
25. More precisely, the condition is that the reference of ξ in $\mathfrak{A}$ – $R_{L,\mathfrak{A}}(\xi)$ – is closed under all automorphisms of $\mathfrak{A}$. For example, consider the logical notion 'exactly one', construed either as a primitive or as a defined logical notion in a language L in a logical framework $\mathscr{L}$. Let $\mathfrak{A}$ be a model for L with a universe $A = \{a_1, a_2, a_3, a_4\}$. Then, $R_{L,\mathfrak{A}}$('exactly one') $= \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}\}$. $R_{L,\mathfrak{A}}$ constitutes a *plenum* in A in the sense that, for any permutation p of A and any $X \in R_{L,\mathfrak{A}}$('exactly one'), the image of X under p is already in $R_{L,\mathfrak{A}}$('exactly one').
26. A more detailed account of the "anatomy" of indeterminacy in logical frameworks (and other frameworks of analogous structure) will center on three principles: (i) the principle of distinguished constants, that is, constants whose interpretation is "hard-wired" into the framework, versus nondistinguished constants; (ii) the invariance principle characterizing the distinguished constants (a principle that determines the kind of distinctions that distinguished constants are capable of making), and (iii) the principle of variability of models [a principle that says what structures of objects (relative to a given language) are represented by models (for the language)]. I have discussed these principles at length elsewhere. [See Sher (1991, 1996a,b).]

27. For example, by adding to our logical framework 'metal' and 'conducts electricity' as extralogical distinguished constants, we rule out the existence of models in which the extension of 'metal' is not included in the extension of 'conducts electricity'.
28. A physical invariance principle will essentially say that distinguished physical constants are invariant under physically equivalent conditions.

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